Exponential Decay and Time Constants

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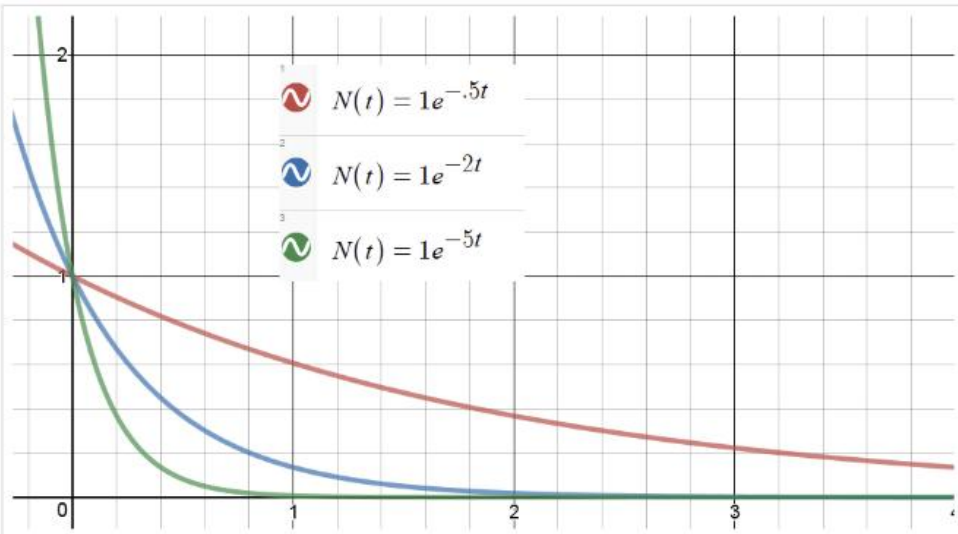
The other day, a colleague showed me a diagram about current vs. time in an RC circuit. The diagram helped me realize that there is another way to think about time constants and exponential decay, a way I had completely overlooked. I had all the right pieces floating around but I had never put them together.

WHAT'S A TIME CONSTANT?

Here are three graphs showing a quantity that is decaying exponentially toward zero:



Each of these is a graph of a function: $N(t) = N_o e^{-kt}$

The only difference is the value of the constant, *k*. Higher values of *k* lead, in a sense, to faster decay.

To help emphasize this, we can define a constant: $\tau = 1/k$

Then we can re-write the function this way: $N(t) = N_o e^{-t/\tau}$

We call τ the "time constant" for this decay. It has the units of time. And it gives us an intuitive feeling for how fast a function is decaying.

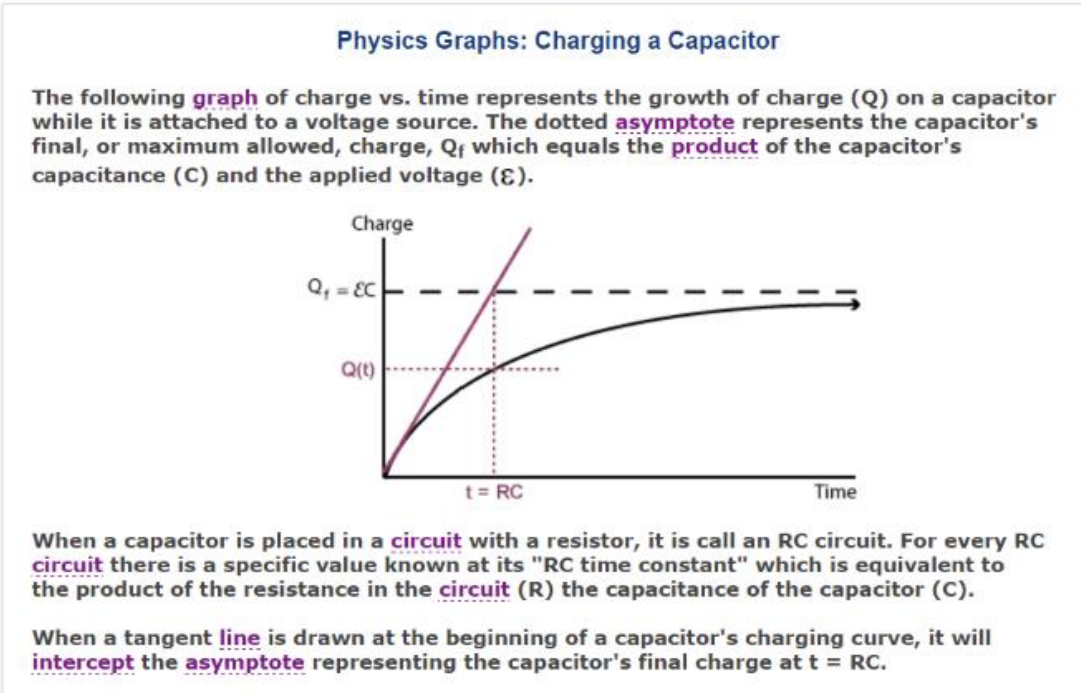
For every time constant that passes, our decaying quantity gets reduced by another factor of e.

So after one time constant has passed, the function's value is N_o/e. After two time constants, it's N_o/e^2. After three, N_o/e^3 …and so on.

But there is another way to think about the time constant.

The time constant tells us how long it would take to reach the asymptote <u>at the current rate of change.</u>

Maybe this should have been obvious, but I never thought about that way until just a few days ago. Here's the diagram that got me started:



http://www.algebralab.org/practice/practice.aspx?file=Reading_ChargingCapacitor.xml

The claim here is made in the context of a charging capacitor, but it is actually true for any function that is approaching an asymptote exponentially. Here is an animation I made in *Desmos* to illustrate the point — you can watch the video here.

In the still picture above, you can see the tangent line at t=0. The dotted green segment on the t-axis represents the time it would take for the tangent line to reach the asymptote. When you run the video, you will see the function decrease and the tangent line get shallower. But that green segment stays the same length.

(If you want to play with the Desmos file, you can get it here :)

WAIT – WHAT'S A TIME CONSTANT NOW?

A quantity that decays exponentially approaches its final value asymptotically. And as it decays, its rate of decay decreases as well. (That's the defining characteristic of exponential decay: the rate of decay toward the final value is proportional to the current distance from that value. In fact, we can write that as a differential equation. I went on and on about that here.)

But now suppose that at some moment, the decay rate were to become constant. Instead of taking forever, the function would now be able to reach its asymptote in a finite amount of time. Not in just any old amount of time, but in fact in an amount that equals the “time constant” of the decay. It doesn't matter what moment you pick. From any starting point during exponential decay, if the rate were to stay constant (which it doesn't, but still…) the time to reach the asymptote would always comes out to $\tau = 1/k$.

Or to say the same thing geometrically: the tangent line from any point on the curve will intercept the asymptote after one time constant has elapsed. That's what the animation is trying to emphasize. I chose a k value of .2 so my time constant was 1/.2 = 5 seconds. If you open the Desmos file, you can change the constants to see what changes and what doesn't.

SO WHY DID THIS SURPRISE ME?

I wasn't making the connection between the exponential decay and the differential equation lurking in the background (even though I teach that connection and have blogged about it).

That the function decreases with a rate of change that is proportional to the value of the function is actually a first-order differential equation. The exponential decay is its solution. (Again, gory detail here.)

$$\text{If } \frac{dN(t)}{dt} = -k \times N(t) \text{ then } N(t) = N_o e^{-kt}.$$

And if slope is rise over run, then run is rise over slope! When the function is starting from a higher value, the tangent has a slope that is **PROPORTIONALLY** higher! So that proportionally higher slope will get you back to the asymptote in the same amount of time every time.

To find how much time we are talking about, divide the “rise” (or in this case, a “fall”) by the slope:

$$\frac{"Fall"}{"slope"} = \frac{-N(t)}{-kN(t)} = \frac{1}{k} = \tau$$

I'll leave a more rigorous, math-y proof as an exercise for those of you in your first calculus class.